Kepler's 3rd Law and the Mass of Jupiter

Adapted from `http://www.astro.lsa.umich.edu/Academics/Undergrad/labs.php`

Introduction

From Chapter 5, Kepler's third law for two bodies orbiting each other is

$$\left[\frac{R}{1\ \mathrm{AU}}\right]^3 = \left[\frac{m_1 + m_2}{1\ M_\odot}\right]\left[\frac{P}{1\ \mathrm{yr}}\right]^2 , \tag{1}$$

where R is the distance between bodies, m_1 and m_2 are the masses of the two bodies, and P is the length of time to complete one orbit. If $m_1 \gg m_2$, what approximation can you make?

In the 1600s, Galileo used a telescope to discover that Jupiter had four moons and made exhaustive studies of this system. The Jupiter system was especially important because it is, gravitationally, a "miniature" solar system. We will use this mini solar system to test Kepler's third law.

Directions

The CLEA software simulates a telescope, with which you "observe" the moons. Open the **CLEA Labs** folder and double click on the CLEA icon: **Moons of Jupiter** and **Login**. After entering your names, choose the **Start** option to set the starting date and time. You will this table later to reset the **Interval Between Observations**.

Jupiter is in the center of the screen, with its moons to either side. The current magnification is displayed in the upper left hand corner of the screen. The date, UT (the time in Greenwich, England) and JD (Julian Date) are displayed in the lower left hand corner.

Click on each moon, using the highest magnification possible, to find the distance between the moon and Jupiter in Jupiter diameters, x. Determine the corresponding uncertainty in x. If the moon is behind Jupiter, record the distance for that moon to be zero. If this is the case, note that your uncertainty will be larger.

For each moon, make a table as follows:

Moon Name		
t (days)	x (j.d.)	δx (j.d.)

At each observation interval record how many days (may be fractional) have elapsed, the position of the moon, and the uncertainty in the position. To be consistent, give any observations east of Jupiter a negative sign and observations west of Jupiter a positive sign. Give sufficient time coverage to all four orbits. For instance, since Io's orbital period is significantly shorter than that of Callisto, you will have to change your observation interval to both get good time coverage and to make efficient use of your observing time. You need to cover a full period of the orbit from, for example, the moon's most eastern position to its most western position *and back*.

1. Using your choice of graphing program, plot each moon's position as a function of time, displaying your uncertainties as vertical error bars.

2. The data should take the shape of a sine curve. Determine the period, P, and the distance from Jupiter, R, for each moon. Estimate your uncertainty in each value of R.

3. You now have 4 sets of (P, R) coordinates, with corresponding uncertainties. Now make yet another plot, with $\ln(R/\mathrm{AU})$ on the y-axis and $\ln(P/\mathrm{yr})$ on the x-axis.

4. Find the slope using equation 3. Find the uncertainty in the calculated slope using equation 6. Show your work. Check your calculation using a program such as Excel. What is the expected value of the slope? Does it agree with your value? What does this test?

5. Find the intercept using equation 2. Show your work. Check your calculation using a program such as Excel. Find the uncertainty in the calculated intercept using equation 5. What is the expected value of the intercept? Does it agree with your value? What does this test?

   Note that the accepted value of M_J is 1.8986×10^{27} kg.

Thinking Questions

6. There is one glaring assumption in our computerized observing method. What is it? How could you get such even time coverage?

Least-Squares Fit to a Straight Line

Adapted from *Data Reduction and Error Analysis for the Physical Sciences*, by Philip R. Bevington and D. Keith Robinson.

Suppose that you take measurements $(x_i,\ y_i,\ \sigma_i)$, where σ_i is the uncertainty in the measured value y_i. Then, the straight line that best approximates this data has slope a and intercept b given by:

$$b = \frac{1}{\Delta}\left(\sum \frac{x_i^2}{\sigma_i^2} \sum \frac{y_i}{\sigma_i^2} - \sum \frac{x_i}{\sigma_i^2} \sum \frac{x_i y_i}{\sigma_i^2}\right) \tag{2}$$

$$a = \frac{1}{\Delta}\left(\sum \frac{1}{\sigma_i^2} \sum \frac{x_i y_i}{\sigma_i^2} - \sum \frac{x_i}{\sigma_i^2} \sum \frac{y_i}{\sigma_i^2}\right) \tag{3}$$

$$\Delta = \sum \frac{1}{\sigma_i^2} \sum \frac{x_i^2}{\sigma_i^2} - \left(\sum \frac{x_i}{\sigma_i^2}\right)^2 \tag{4}$$

Furthermore, the uncertainty in these values is given by:

$$\sigma_b = \sqrt{\frac{1}{\Delta} \sum \frac{x_i^2}{\sigma_i^2}} \tag{5}$$

$$\sigma_a = \sqrt{\frac{1}{\Delta} \sum \frac{1}{\sigma_i^2}} \tag{6}$$